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Mass transport of interfacial waves in a two-layer fluid system

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Effects of viscous damping on mass transport velocity in a two-layer fluid system are studied. A temporally decaying small-amplitude interfacial wave is assumed to propagate in the fluids. The establishment and the decay of mean motions are considered as an initial-boundary-value problem. This transient problem is solved by using a Laplace transform with a numerical inversion. It is found that thin 'second boundary layers' are formed adjacent to the interfacial Stokes boundary layers. The thickness of these second boundary layers is of $O(\epsilon^{1/2})$ in the non-dimensional form, where ϵ is the dimensionless Stokes boundary layer thickness defined as $\epsilon = \hat{k}\hat{\delta} = \hat{k}\left(2\hat{\nu}/\hat{\sigma}\right)^{1/2}$ for an interfacial wave with wave amplitude $\hat{a}$, wavenumber $\hat{k}$ and frequency $\hat{\sigma}$ in a fluid with viscosity $\hat{\nu}$. Inside the second boundary layers there exists a strong steady streaming of $O(\alpha^2\epsilon^{-1/2})$, where $\alpha = \hat{k}\hat{a}$ is the surface wave slope. The mass transport velocity near the interface is much larger than that in a single-layer system, which is $O(\alpha^2)$ (e.g. Longuet-Higgins 1953; Craik 1982). In the core regions outside the thin second boundary layers, the mass transport velocity is enhanced by the diffusion of the mean interfacial velocity and vorticity. Because of vertical diffusion and viscous damping of the mean interfacial vorticity, the 'interfacial second boundary layers' diminish as time increases. The mean motions eventually die out owing to viscous attenuation. The mass transport velocity profiles are very different from those obtained by Dore (1970, 1973) which ignored viscous attenuation. When a temporally decaying small-amplitude surface progressive wave is propagating in the system, the mean motions are found to be much less significant, $O(\alpha^2)$.

1. Introduction

Mass transport, a steady Lagrangian current, is generated by wave motions and is important in determining the migration of pollutant in fluid and the transport of sediment near the seabed. The mass transport velocity of small-amplitude interfacial waves in a two-layer fluid system was first calculated by Dore (1970) using the method of matched asymptotic expansions. The fluid system consists of two immiscible fluids with different densities and viscosities. Dore introduced boundary layers in the neighbourhood of the interface, whose thicknesses were assumed to be much larger than the wave amplitude, i.e. $\hat{\delta} \gg \hat{a}$. Using a Cartesian coordinate system, he found that the mass transport velocity was of $O(\hat{k}\hat{a}^2/\hat{\delta})$, where $\hat{k}$ is the wavenumber. Hence, the mass transport in the interfacial waves is much greater than that in a single-layer fluid system (Longuet-Higgins 1953), which is of $O(\hat{k}^2\hat{a}^2)$. Of course Dore's approach breaks down if $\hat{a} \to \hat{\delta}$. Later, Dore (1973) re-examined this problem by using the curvilinear coordinate system originally introduced by Longuet-Higgins (1953) to describe the motions in the interfacial boundary layers, but the conclusion remained the same. The second restrictive assumption used in both of Dore's papers is that the viscous attenuation can be ignored. Although, the effects of the wave amplitude and viscous damping on mass transport have been studied by many researchers for a single-layer system (e.g. Liu & Davis 1977; Craik 1982), no study has been performed for interfacial waves. It is not clear what is the order of magnitude of the mass transport velocity in a two-layer fluid system.

In this paper, the effects of viscous damping on the mass transport velocity are studied for a two-layer fluid system with a two-dimensional small-amplitude interfacial progressive wave. The wave motions are assumed to decay slowly in time because of viscosity. Both fluids are assumed to be immiscible, and a discontinuity of viscosity and density exists on the interface. Perturbation solutions in terms of wave slope $\alpha = \hat{k}\hat{a}$ via a boundary layer approach are sought. The thicknesses of the Stokes boundary layers adjacent to the free surface, the interface, and the bottom boundary are assumed to be of the same order of magnitude as the wave amplitude, i.e. $O(\hat{\delta}) = O(\hat{a})$. For wave motions with frequency $\hat{\sigma}$ in a fluid with viscosity $\hat{\nu}$, the Stokes boundary layer thickness is defined as $\hat{\delta} = (2\hat{\nu}/\hat{\sigma})^{1/2}$. The curvilinear coordinate system of Longuet-Higgins (1953) is adopted to describe the fluid motions inside the Stokes boundary layers. To consider the on-set of the mean motions in the core regions outside the Stokes boundary layers, an initial-boundary-value problem is formulated and is solved by using a Laplace transform with a numerical inversion. Owing to the balance between diffusion and inertia forces the mean velocity (and vorticity) is confined inside the thin layers of $O(\epsilon^{1/2})$ adjacent to the interface, where $\epsilon = \hat{k}\hat{\delta}$, and a strong steady streaming of $O(\alpha^2\epsilon^{-1/2})$ exists. These 'second boundary layers' are similar to the outer layers reported by Stuart (1963, 1966). But the laters are generated by the balance between viscous diffusion and convective inertia, and exist only when the steady-drift Reynolds number is much greater than 1, i.e. $R_s \propto (\alpha/\epsilon)^2 \gg 1$. Unlike Stuart's outer layers, which are steady layers of $O(\epsilon/\alpha)$, the second boundary layers found in present study are unsteady, and their thickness is determined by viscous attenuation and viscous diffusion. The viscous attenuation is assumed to be of the same order of magnitude as the wave slope i.e. $O(\epsilon) = O(\alpha)$. This corresponds to $R_s = O(1)$. But in the case of progressive waves, convection nonlinearity for the mean motions vanishes, so that the steady-drift Reynolds number R_s does not play a role. The mean motion begins to develop after the establishment of the Stokes boundary layers near the free surface, the interface and the bottom boundary due to viscosity. Because of vertical diffusion and viscous attenuation of the mean interfacial vorticity, the thin secondary boundary layers diminish as time increases. The mass transport velocity in the system eventually dies out after a long time. The mean motions of a surface progressive wave propagating in a two-layer fluid system are also examined. The leading-order mean velocity is found to be of $O(\alpha^2)$, much smaller than that of an interfacial wave, $O(\alpha^2\epsilon^{-1/2})$. The mean velocity profiles are similar to that of mean motions of $O(\alpha^2)$ of an interfacial progressive. Second boundary layers exist adjacent to the Stokes boundary layers.

In §2 the formulation of interfacial wave motions in a two-layer fluid system is given. The leading-order solutions including the irrotational motions in the core regions and the associated rotational motions in the boundary layers have been obtained by Dalrymple & Liu (1978). For completeness, these results and the viscous damping coefficient are briefly summarized in §3. The governing equations and boundary conditions for the mean Eulerian drift velocity inside the Stokes boundary layers as well as in the core regions are presented in §4. The solution procedures for obtaining the mass transport in the core regions are also described. The solutions are first compared with those obtained by Craik (1982) for a single-layer fluid system. In this special case the mean velocity is of $O(\alpha^2)$. Excellent agreement between Craik's solutions and the present results is obtained. Numerical results are then given for two-layer fluid systems with or without mean pressure gradients. In both cases numerical results show clearly the existence of the second boundary layers adjacent to the Stokes boundary layers. The magnitude of the mass transport is indeed of $O(\alpha^2\epsilon^{-1/2})$. Finally the mass transport velocity of a surface wave is investigated, and some numerical results are presented.

2. Formulation

Consider a two-dimensional progressive interfacial wave train propagating in a two-layer fluid system with a discontinuity of density and viscosity on the interface. The positive $\hat{x}$-axis is in the direction of wave propagation and the positive $\hat{z}$-axis is pointing upwards from the mean level of the interface. The mean water depths for each layer are $\hat{h}^{(r)}$, $r = 1, 2$. The density of the upper fluid layer, $\hat{\rho}^{(1)}$, is smaller than that in the lower layer, i.e. $\hat{\rho}^{(1)} < \hat{\rho}^{(2)}$, and they are of the same order of magnitude, i.e. $\hat{\rho}^{(2)}/\hat{\rho}^{(1)} = O(1)$. It is also assumed that the wave motion is a stable laminar flow. The wave motion is periodic in time with a frequency $\hat{\sigma}$ and in the $\hat{x}$-direction with a wavenumber $\hat{k}$. The leading-order interfacial displacement, $\hat{\zeta}$, and the leading-order free surface displacement, $\hat{\xi}$, are assumed to be sinusoidal in shape and of the same order of magnitude.

Using the Cartesian coordinates $(\hat{x}, \hat{z})$, we write the equations of motion as

$$\frac{\partial \hat{\boldsymbol{q}}^{(r)}}{\partial \hat{t}} + (\hat{\boldsymbol{q}}^{(r)} \cdot \hat{\nabla})\hat{\boldsymbol{q}}^{(r)} = -\frac{1}{\hat{\rho}^{(r)}}\hat{\nabla}\hat{p}^{(r)} + \hat{\nu}^{(r)}\hat{\nabla}^2\hat{\boldsymbol{q}}^{(r)}, \tag{2.1}$$

$$\hat{\nabla} \cdot \hat{\boldsymbol{q}}^{(r)} = 0, \tag{2.2}$$

where the superscript $r = 1$ or 2 denotes the variables in the upper or the lower layer, t is time, $\hat{\boldsymbol{q}}^{(r)} = (\hat{u}^{(r)}, \hat{w}^{(r)}), \hat{p}^{(r)}, \hat{\rho}^{(r)}$ and $\hat{\nu}^{(r)}$ denote fluid velocity, dynamic pressure, density and kinematic viscosity, in each fluid layer, respectively. A stream function $\hat{\psi}^{(r)}$ is defined such that

$$\hat{u}^{(r)} = \frac{\partial \hat{\psi}^{(r)}}{\partial \hat{z}}, \qquad \hat{w}^{(r)} = -\frac{\partial \hat{\psi}^{(r)}}{\partial \hat{x}}, \tag{2.3}$$

which satisfies the continuity equation, (2.2). By adopting non-dimensional variables according to the following normalizing schemes:

$$(x,z) = \hat{k}(\hat{x},\hat{z}),\ t = \hat{\sigma}\hat{t},\ \psi^{(r)} = (\hat{k}^2/\hat{\sigma})\hat{\psi}^{(r)},\ p^{(r)} = (\hat{k}^2/\hat{\sigma}^2\hat{\rho})\hat{p}^{(r)},\ \boldsymbol{q}^{(r)} = (\hat{k}/\hat{\sigma})\hat{\boldsymbol{q}}^{(r)}, \tag{2.4}$$

one can show that from the momentum equation, (2.1), the non-dimensional stream function satisfies

$$\left(\frac{\partial}{\partial t} + \frac{\partial \psi^{(r)}}{\partial z}\frac{\partial}{\partial x} - \frac{\partial \psi^{(r)}}{\partial x}\frac{\partial}{\partial z}\right)\nabla^2\psi^{(r)} = (1/2)\epsilon^{(r)2}\nabla^4\psi^{(r)}, \tag{2.5}$$

where $\nabla^2\psi^{(r)} = \omega^{(r)}$ represents the vorticity, and $\epsilon^{(r)} = (2\hat{k}^2\hat{\nu}^{(r)}/\hat{\sigma})^{1/2}$ denotes the dimensionless Stokes boundary thickness. The non-dimensional forms of (2.1) and (2.2) are

$$\frac{\partial \boldsymbol{q}^{(r)}}{\partial t}+(\boldsymbol{q}^{(r)}\cdot\nabla)\boldsymbol{q}^{(r)}=-\nabla p^{(r)}+(1/2)\epsilon^{(r)2}\nabla^2\boldsymbol{q}^{(r)}, \tag{2.6}$$

$$\nabla\cdot\boldsymbol{q}^{(r)}=0. \tag{2.7}$$

In most practical problems and laboratory studies the boundary layer thickness is much smaller than the typical wavelength, i.e. $\epsilon^{(r)} \ll 1$. Therefore, flow motions are amenable to a boundary layer analysis. The whole fluid domain is divided into (i) the Stokes boundary layer regions of thickness $O(\epsilon^{(r)})$ adjacent to the free surface, interface, and bottom boundary and (ii) the core regions, exterior to these layers. The oscillatory vorticity is confined to those boundary layers. The boundary layer thickness in the oscillatory flows should also be much less than the corresponding fluid depth ($\epsilon^{(r)} \ll \hat{k}\hat{h}^{(r)}$).

Because the boundary layer thickness is of the same order of magnitude as the wave amplitude, $\epsilon \leqslant \alpha$, and the free surface and interface are moving, we use the orthogonal curvilinear coordinate system (s,n) introduced by Longuet-Higgins (1953) to describe the motions of fluids in the boundary layers adjacent to the free surface and interface, where s is tangential to the free surface or to the interface and n is normal to s pointing downwards. In terms of the curvilinear coordinates the velocities in the boundary layers can be expressed as

$$u^{(r)}=\frac{\partial\psi^{(r)}}{\partial n},\qquad w^{(r)}=-\frac{1}{\eta}\frac{\partial\psi^{(r)}}{\partial s}, \tag{2.8}$$

where

$$\eta=1-n\kappa,\quad \Omega=\left(-\frac{1}{\eta}\frac{\partial^2\psi^{(r)}}{\partial s^2}+\kappa\frac{\partial\psi^{(r)}}{\partial n}\right)\Bigg|_{n=0},\quad \frac{\partial\kappa}{\partial t}=\frac{\partial\Omega}{\partial s}.$$

The dimensionless governing equation of the motions in the Stokes boundary layers can be written as, in terms of the curvilinear coordinates (Longuet-Higgins 1953; Dore 1978),

$$\left(\frac{\partial}{\partial t}+\dot{s}^{(r)}\frac{\partial}{\partial s}+\dot{n}^{(r)}\frac{\partial}{\partial n}\right)\nabla^2\psi^{(r)}=\tfrac{1}{2}\epsilon^{(r)2}\nabla^4\psi^{(r)}, \tag{2.9}$$

$$\nabla^2\equiv\frac{1}{\eta}\left[\frac{\partial}{\partial s}\left(\frac{1}{\eta}\frac{\partial}{\partial s}\right)+\frac{\partial}{\partial n}\left(\eta\frac{\partial}{\partial n}\right)\right], \tag{2.10}$$

$$\dot{s}^{(r)}=\frac{1}{\eta}\left(\frac{\partial\psi'^{(r)}}{\partial n}+n\Omega\right),\qquad \dot{n}^{(r)}=-\frac{1}{\eta}\frac{\partial\psi'^{(r)}}{\partial s}, \tag{2.11}$$

where $\psi'^{(r)}=\psi^{(r)}-\psi^{(b)}$, and $\psi^{(b)}$ is the stream function for the motion of the boundary surface itself ($n=0$).

Because wave motions decay slowly in time owing to viscosity and the development of second-order mean motions in core regions also depends on viscosity, there are two time scales involved in the flow motions: (i) the short time scale t for the oscillatory motions and (ii) the long time scale $T=\beta t$, where β is viscous damping rate of $O(\epsilon^{(r)})$ (see § 3). Therefore, we can write the time derivative as follows:

$$\partial/\partial t\rightarrow\partial/\partial t+\beta\partial/\partial T. \tag{2.12}$$

For small-amplitude wave motions the wave slope is a small parameter, i.e. $\alpha \ll 1$. In the present analysis, it is assumed that the wave amplitude is of the same order of magnitude as those of the Stokes boundary layer thickness in each fluid layer, i.e. $O(\alpha) = O(\epsilon^{(1)}) = O(\epsilon^{(2)}) \ll 1$. The leading-order free surface displacement is written as

$$\zeta = \alpha A(T) \exp[\mathrm{i}(x - t)] + O(\alpha^2), \tag{2.13}$$

where $A(T)$ is the wave amplitude decaying slowly in time, and is to be determined.

3. The leading-order wave motions

To $O(\alpha)$, the effects of viscosity can be neglected in most of the flow domain except in the Stokes boundary layers adjacent to the free surface, interface and bottom boundary (e.g. Mei & Liu 1973). Outside the Stokes boundary layers flows are irrotational. Thus, we can write the leading-order stream function $\psi_1^{(r)}$ as

$$\psi_1^{(r)} = \varphi_1^{(r)} + \chi_1^{(r)}, \tag{3.1}$$

where $\varphi_1^{(r)}$ and $\chi_1^{(r)}$ are irrotational and rotational stream functions respectively, and $\chi_1^{(r)}$ exists only inside the boundary layers.

3.1. *The potential solutions*

The governing equation for the irrotational stream function φ_1 is simply

$$\nabla^2 \varphi_1^{(r)} = 0. \tag{3.2}$$

The solutions of $\varphi_1^{(r)}$ for a progressive wave propagating in a two-layer fluid system are readily obtained with appropriate boundary conditions. The details of the solution procedures are given in Dalrymple & Liu (1978). In terms of the present notation, the irrotational stream functions in each layer are respectively

$$\varphi_1{}^{(1)} = \alpha A(T) \left[\cosh(z - h^{(1)}) + (\hat{g}\hat{k}/\hat{\sigma}^2)\sinh(z - h^{(1)})\right] \exp[\mathrm{i}(x - t)], \tag{3.3}$$

$$\varphi_1{}^{(2)} = \alpha A(T) c_1 \sinh(z + h^{(2)}) \exp[\mathrm{i}(x - t)], \tag{3.4}$$

where $\hat{g}$ is the gravitational acceleration, and c_1 is a constant determined by fluid properties and wave parameters (see Appendix A). The dispersion relation and the corresponding interfacial displacement are also presented in Appendix A.

3.2. *The boundary layer corrections*

Substituting (3.1) into (2.5) or (2.9), employing (3.2) and neglecting higher-order terms, we obtain, in the Stokes boundary layers,

$$\frac{\partial}{\partial t}\nabla^2 \chi_1^{(r)} = (1/2)\epsilon^{(r)2}\nabla^4 \chi_1^{(r)}, \tag{3.5}$$

for each layer of fluid. In the boundary layers the tangential derivative of the rotational velocity is much smaller than its normal derivative. The rotational velocity vanishes outside the boundary layers. Therefore, the governing equations of rotational flows in the Stokes boundary layers adjacent to the free surface and interface can be simplified as

$$\frac{\partial \chi_1^{(r)}}{\partial t} = (1/2)\epsilon^{(r)2}\frac{\partial^2 \chi_1^{(r)}}{\partial n^2}. \tag{3.6}$$

Similarly, we have in the bottom boundary layer, in terms of the Cartesian coordinates,

$$\frac{\partial \chi_1^{(2)}}{\partial t} = (1/2)\epsilon^{(2)2}\frac{\partial^2 \chi_1^{(2)}}{\partial z^2}. \tag{3.7}$$

The boundary conditions and the solutions for each boundary layer are discussed in the following sections.

3.2.1. *The free surface boundary layer*

For the free surface Stokes boundary layers the curvilinear coordinates, (s, n), are used with s being tangential to the free surface and n normal to s pointing into the upper layer, i.e.

$$x = -s + O(\alpha), \quad z = (h^{(1)} - n) + O(\alpha).$$

Since the surface is not contaminated, the tangential stress along the free surface must vanish, i.e. correct to $O(\alpha)$

$$\left.\frac{\partial^2 \chi_1^{(1)}}{\partial n^2}\right|_{n=0} = -\left(\frac{\partial^2 \varphi_1^{(1)}}{\partial z^2} - \frac{\partial^2 \varphi_1^{(1)}}{\partial x^2}\right)_{z=h^{(1)}}, \tag{3.8}$$

where the values of irrotational quantities on the free surface ($n = 0$) have been replaced by those on the mean level of the free surface ($z = h^{(1)}$), correct to $O(\alpha)$. The rotational flow of $O(\alpha)$ diminishes just beyond the boundary layer so that

$$\frac{\partial \chi_1^{(1)}}{\partial n} \to 0 \quad \text{as } n/\epsilon^{(1)} \to \infty. \tag{3.9}$$

Integrating (3.6) and using (3.8) and (3.9), we obtain (e.g. Liu 1977)

$$\frac{\partial \chi_1^{(1)}}{\partial n} = \alpha(1+\mathrm{i})A(T)\epsilon^{(1)} \exp\left[-(1-\mathrm{i})n/\epsilon^{(1)}\right] \exp\left[-\mathrm{i}(s+t)\right]. \tag{3.10}$$

3.2.2. *The interfacial boundary layers*

In the Stokes boundary layers adjacent to the interface, s is tangential to the interface and n is normal to s pointing into the lower layer, i.e.

$$x = -s + O(\alpha), \quad z = -n + O(\alpha).$$

On the interface, $n = 0$, the tangential velocity and tangential stress are continuous. Thus

$$\left.\frac{\partial \chi_1^{(1)}}{\partial n}\right|_{n=0} - \left.\frac{\partial \varphi_1^{(1)}}{\partial z}\right|_{z=0} = \left.\frac{\partial \chi_1^{(2)}}{\partial n}\right|_{n=0} - \left.\frac{\partial \varphi_1^{(2)}}{\partial z}\right|_{z=0}, \tag{3.11}$$

$$\left.\frac{\partial^2 \chi_1^{(1)}}{\partial n^2}\right|_{n=0} = \left(\frac{\hat{\rho}^{(2)}\hat{\nu}^{(2)}}{\hat{\rho}^{(1)}\hat{\nu}^{(1)}}\right) \left.\frac{\partial^2 \chi_1^{(2)}}{\partial n^2}\right|_{n=0}, \tag{3.12}$$

where contributions from the irrotational solutions to the tangential stresses are of higher order, in terms of $\epsilon^{(r)}$, and are neglected, and the values of the irrotational quantities on the interface are replaced by those on the mean level of the interface. At the outer edges of the interfacial boundary layers the rotational velocities vanish. Therefore

$$\frac{\partial \chi_1^{(1)}}{\partial n} \to 0 \quad \text{as } n/\epsilon^{(1)} \to -\infty, \qquad \frac{\partial \chi_1^{(2)}}{\partial n} \to 0 \quad \text{as } n/\epsilon^{(2)} \to \infty. \tag{3.13}$$

The solutions for the rotational stream functions in the interfacial boundary layers can be obtained by integrating (3.6) with boundary conditions (3.11), (3.12) and (3.13), and can be expressed as (Dalrymple & Liu 1978)

$$\frac{\partial \chi_1^{(1)}}{\partial n} = \alpha c_2^{(1)} A(T) \exp\left[(1-\mathrm{i})n/\epsilon^{(1)}\right] \exp\left[-\mathrm{i}(s+t)\right], \tag{3.14}$$

$$\frac{\partial \chi_1^{(2)}}{\partial n} = -\alpha c_2^{(2)} A(T) \exp\left[-(1-\mathrm{i})n/\epsilon^{(2)}\right] \exp\left[-\mathrm{i}(s+t)\right], \tag{3.15}$$

where $c_2^{(1)}$ and $c_2^{(2)}$ are constants, and their expressions can be found in Appendix A.

3.2.3. *The bottom boundary layer*

At the bottom ($z = -h^{(2)}$), the rotational velocity exists to satisfy no-slip condition, i.e.

$$\frac{\partial \chi_1^{(2)}}{\partial z} + \frac{\partial \varphi_1^{(2)}}{\partial z} = 0. \tag{3.16}$$

Just beyond the boundary layer, the rotational velocity vanishes up to $O(\alpha)$. Hence

$$\frac{\partial \chi_1^{(2)}}{\partial z} \to 0 \quad \text{as } (z + h^{(2)})/\epsilon^{(2)} \to \infty. \tag{3.17}$$

Upon integrating (3.7) with boundary conditions (3.16) and (3.17), it is found that in this Stokes boundary layer (Dalrymple & Liu 1978)

$$\frac{\partial \chi_1^{(2)}}{\partial z} = -\alpha A(T) c_1 \exp[-(1-\mathrm{i})(z+h^{(2)})/\epsilon^{(2)}] \exp\left[\mathrm{i}(x-t)\right]. \tag{3.18}$$

3.3. *Viscous damping factor*

By considering the energy balance in the two-layer fluid system, Dalrymple & Liu (1978) found that the temporally decaying factor is

$$A(T) = \mathrm{e}^{-T}, \qquad T = \beta t, \tag{3.19}$$

where the non-dimensional decaying rate, β, is

$$\beta = \frac{\epsilon^{(1)}}{4}\left(\frac{\hat{\sigma}^2}{\hat{g}\hat{k}}\right) \frac{c_2^{(1)2} + (\hat{\rho}^{(2)}/\hat{\rho}^{(1)})(\epsilon^{(2)}/\epsilon^{(1)})(c_2^{(2)2} + c_1^2)}{1 + [\hat{\rho}^{(2)}/\hat{\rho}^{(1)} - 1](\hat{g}\hat{k}/\hat{\sigma}^2)^2 \cosh^2 h^{(1)} [(\hat{\sigma}^2/\hat{g}\hat{k}) - \tanh h^{(1)}]^2} + O(\epsilon^2). \tag{3.20}$$

The main contributions to the energy dissipation in a two-layer fluid system come from the Stokes boundary layers adjacent to the bottom and interface, $O(\epsilon^{(r)})$. The dissipation in the free surface boundary layer and the core regions is of $O(\epsilon^{(1)3})$ and $O(\epsilon^{(r)2})$, respectively (Mei & Liu 1973). Because $c_1, c_2^{(1)}$ and $c_2^{(2)}$ are quantities of $O(1)$, at leading order β is of $O(\epsilon^{(r)})$.

4. Mean Eulerian drift velocity

Consider the on-set of motions in a two-layer fluid system. The first-order irrotational wave motions are established instantaneously, and, after a few wave periods, the oscillatory Stokes boundary layers are formed. The time needed for the establishment of the first-order motions is just several wave periods. The mean motions in the core regions are induced by diffusion and advection of the mean velocity and/or vorticity at the outer edges of the Stokes boundary layers. In the case of a progressive interfacial wave, the mean motion is expected to be unidirectional, so that the mean velocity and/or vorticity penetrate into the core regions only through viscous diffusion, which has a much larger time scale than a wave period. Therefore we can assume that the mean motions in core regions begin to develop just after the first-order wave motions, and the mean velocity and/or vorticity at the outer edges of the Stokes boundary layers are fully established.

4.1. *The Stokes boundary layers*

Because β is of $O(\epsilon^{(r)})$ (see (3.20)) and within the Stokes boundary layers

$$\frac{\partial}{\partial n} = O(1/\epsilon^{(r)}), \qquad \frac{\partial}{\partial s} = O(1), \qquad \frac{\partial}{\partial T} = O(1),$$

the time-averaged (over a wave period) equation of motion in the Stokes boundary layers, in terms of the curvilinear coordinates, can be expressed as (Longuet-Higgins 1953; Dore 1978)

$$\tfrac{1}{2}\epsilon^{(r)2}\frac{\partial^2\overline{\varpi}^{(r)}}{\partial n^2} = \overline{\left(\frac{\partial\psi_1'^{(r)}}{\partial n}\frac{\partial}{\partial s} - \frac{\partial\psi_1'^{(r)}}{\partial s}\frac{\partial}{\partial n}\right)\omega_1^{(r)}}, \tag{4.1}$$

where the bar signifies that a quantity is averaged over a wave period, and the first-order oscillatory vorticity and the mean vorticity are

$$\omega_1^{(r)} = \frac{\partial^2\chi_1^{(r)}}{\partial n^2}, \qquad \overline{\varpi}^{(r)} = \frac{\partial^2\overline{\psi}^{(r)}}{\partial n^2} - \kappa_1\overline{\frac{\partial\psi_1^{(r)}}{\partial n}}, \tag{4.2}$$

respectively, with the curvature of the boundary $\kappa_1 = \mathrm{i}\partial\psi_1^{(r)}/\partial s$. In the following subsections, we specify the boundary conditions for the mean drift field in each boundary layer. Solutions inside the boundary layers are then presented.

4.1.1. *The free surface boundary layer*

Without any contamination on the free surface, the mean tangential stress also vanishes on the free surface. Hence, for a progressive wave we have (Longuet-Higgins 1953)

$$\overline{\varpi}^{(1)} + 2\kappa_1\overline{\frac{\partial\psi_1^{(1)}}{\partial n}} = 0 \quad \text{on } n = 0. \tag{4.3}$$

Integrating the boundary layer equation (4.1) with the first-order solutions (3.3) and (3.10), we obtain

$$\overline{\varpi}^{(1)} = \alpha^2A^2(T)\left(\frac{\hat{g}\hat{k}}{\hat{\sigma}^2}\right)[2-(1-\mathrm{i})\theta_1\exp[-(1-\mathrm{i})\theta_1]-\exp[-(1-\mathrm{i})\theta_1]], \tag{4.4}$$

where $\theta_1 = n/\epsilon^{(1)}$. At the outer edge of the free surface boundary layer, $\theta_1 \to \infty$, the second-order mean Eulerian vorticity is

$$\overline{\varpi}^{(1)} = 2\alpha^2A^2(T)\left(\frac{\hat{g}\hat{k}}{\hat{\sigma}^2}\right). \tag{4.5}$$

In terms of the Cartesian coordinate system, this vorticity is, correct to $O(\alpha^2)$, equivalent to

$$\frac{\partial\overline{u}^{(1)}}{\partial z} = 2\alpha^2A^2(T)\left(\frac{\hat{g}\hat{k}}{\hat{\sigma}^2}\right) \quad \text{on } z = h^{(1)}. \tag{4.6}$$

This mean vorticity will penetrate into the core regions through diffusion and convection, and (4.6) serves as a boundary condition for the mean flow motion in the core region.

4.1.2. *The interfacial boundary layers*

On the interface, the mean Eulerian drift velocity and the mean tangential stress are continuous. Thus the following interfacial boundary conditions must be satisfied by the mean quantities,

$$\frac{\partial \overline{\psi}^{(1)}}{\partial n} = \frac{\partial \overline{\psi}^{(2)}}{\partial n} \quad \text{on } n = 0, \tag{4.7}$$

and

$$\left(\frac{\hat{\rho}^{(1)}\hat{\nu}^{(1)}}{\hat{\rho}^{(2)}\hat{\nu}^{(2)}}\right)\overline{\omega}^{(1)} = \overline{\omega}^{(2)} \quad \text{on } n = 0. \tag{4.8}$$

Upon integrating the boundary layer equation (4.1) in the two interfacial boundary layers respectively, the mean Eulerian drift velocities in the interfacial boundary layers can be expressed as

$$\begin{aligned}\frac{\partial \overline{\psi}^{(1)}}{\partial n} = & -\alpha^2 A^2(T) c_2^{(1)} \left\{ \frac{(1-\mathrm{i})}{2} u_0^{(1)} \theta_2 \exp[(1-\mathrm{i})\theta_2] - u_0^{(1)} \exp[(1-\mathrm{i})\theta_2] \right. \\ & \left. + \tfrac{1}{4} c_2^{(1)} \exp(2\theta_2) - \frac{\mathrm{i}}{2} c_2^{(1)} \exp[(1-\mathrm{i})\theta_2] \right\} + \overline{\kappa_1 \varphi_1^{(1)}}\Big|_{z=0} \\ & + d_1^{(1)} n + d_2^{(1)} + O(\alpha^2 \epsilon^{(1)}),\end{aligned} \tag{4.9}$$

$$\begin{aligned}\frac{\partial \overline{\psi}^{(2)}}{\partial n} = & -\alpha^2 A^2(T) c_2^{(2)} \left\{ \frac{(1-\mathrm{i})}{2} u_0^{(2)} \theta_3 \exp[-(1-\mathrm{i})\theta_3] + u_0^{(2)} \exp[-(1-\mathrm{i})\theta_3] \right. \\ & \left. + \tfrac{1}{4} c_2^{(2)} \exp(-2\theta_3) - \frac{\mathrm{i}}{2} c_2^{(2)} \exp[-(1-\mathrm{i})\theta_3] \right\} + \overline{\kappa_1 \varphi_1^{(2)}}\Big|_{z=0} \\ & + d_1^{(2)} n + d_2^{(2)} + O(\alpha^2 \epsilon^{(2)}),\end{aligned} \tag{4.10}$$

where $\theta_2 = n/\epsilon^{(1)}$, $\theta_3 = n/\epsilon^{(2)}$, and the $u_0^{(r)}$ are constants from the first-order solutions (see Appendix A). $d_1^{(r)}$ and $d_2^{(r)}$ are constants to be determined from the interfacial boundary conditions and by matching the boundary layer solutions with those in the core regions. From (4.9) and (4.10) the mean Eulerian drift velocities at the outer edges of the interfacial boundary layers are

$$\frac{\partial \overline{\psi}^{(1)}}{\partial n} = \overline{\kappa_1 \varphi_1^{(1)}}\Big|_{z=0} + d_1^{(1)} n + d_2^{(1)} + O(\alpha^2 \epsilon^{(1)}), \tag{4.11}$$

$$\frac{\partial \overline{\psi}^{(2)}}{\partial n} = \overline{\kappa_1 \varphi_1^{(2)}}\Big|_{z=0} + d_1^{(2)} n + d_2^{(2)} + O(\alpha^2 \epsilon^{(2)}), \tag{4.12}$$

respectively. Equations (4.11) and (4.12) imply that as $z \to 0$ the mean Eulerian drift velocities in core regions of the top and bottom layers, correct to $O(\alpha^2)$, approach asymptotically,

$$\overline{u}^{(1)} = d_1^{(1)} z - d_2^{(1)} - \overline{\kappa_1 \varphi_1^{(1)}}\Big|_{z=0} + O(\alpha^2 \epsilon^{(1)}), \tag{4.13}$$

$$\overline{u}^{(2)} = d_1^{(2)} z - d_2^{(2)} - \overline{\kappa_1 \varphi_1^{(2)}}\Big|_{z=0} + O(\alpha^2 \epsilon^{(2)}), \tag{4.14}$$

respectively.

Substituting (4.9) and (4.10) into (4.7) and (4.8), and employing (4.13) and (4.14), one can readily show that

$$\left(\overline{u}^{(2)} - \overline{u}^{(1)}\right)\Big|_{z=0} = -\left(d_2^{(2)} - d_2^{(1)}\right) - \overline{\kappa_1\left(\varphi_1^{(2)} - \varphi_1^{(1)}\right)}\Big|_{z=0} + O(\alpha^2\epsilon^{(r)})$$
$$= -\alpha^2 A^2(T)\left[c_2^{(2)}\left(u_0^{(2)} + \tfrac{1}{4}c_2^{(2)}\right) + c_2^{(1)}\left(u_0^{(1)} - \tfrac{1}{4}c_2^{(1)}\right)\right] + O(\alpha^2\epsilon^{(r)}), \quad (4.15)$$

and

$$\left[\frac{\partial\overline{u}^{(2)}}{\partial z} - \left(\frac{\hat{\rho}^{(1)}\hat{\nu}^{(1)}}{\hat{\rho}^{(2)}\hat{\nu}^{(2)}}\right)\frac{\partial\overline{u}^{(1)}}{\partial z}\right]\Bigg|_{z=0} = d_1^{(2)} - \left(\frac{\hat{\rho}^{(1)}\hat{\nu}^{(1)}}{\hat{\rho}^{(2)}\hat{\nu}^{(2)}}\right)d_1^{(1)}$$
$$= -\tfrac{1}{2}\alpha^2 A^2(T)\frac{c_2^{(2)}}{\epsilon^{(2)}}\left(u_0^{(2)} - u_0^{(1)}\right) + O(\alpha^2). \quad (4.16)$$

Equations (4.15) and (4.16) represent the mean velocity and stress jumps, respectively, across the interfacial Stokes boundary layers. When the viscous damping is neglected, these results agree with those obtained by Dore (1973). Equation (4.16) demonstrates that at the outer edges of the interfacial Stokes boundary layers the mean velocity gradient is of $O(\alpha^2/\epsilon^{(2)})$.

4.1.3. *The bottom boundary layer*

In the bottom boundary layer, the Cartesian coordinate system (x, z) can be used. By requiring the no-slip condition at the bottom for the mean motions, i.e.

$$\frac{\partial\overline{\psi}^{(2)}}{\partial z} = 0 \quad \text{on } z = -h^{(2)}, \quad (4.17)$$

and

$$\frac{\partial\overline{\psi}^{(2)}}{\partial z} \to \text{finite} \quad \text{as } (z + h^{(2)})/\epsilon \to \infty, \quad (4.18)$$

and integrating (4.1) the mean Eulerian drift velocity is obtained in the bottom boundary layer:

$$\frac{\partial\overline{\psi}^{(2)}}{\partial z} = \tfrac{1}{2}\alpha^2 c_1^2 A^2(T)\Big\{-(1-\mathrm{i})\theta_4\exp[-(1-\mathrm{i})\theta_4]$$
$$+(2+\mathrm{i})\Big(1-\exp[-(1-\mathrm{i})\theta_4]\Big) + \tfrac{1}{2}\Big(\exp[-(1-\mathrm{i})\theta_4]-1\Big)\Big\}, \quad (4.19)$$

where $\theta_4 = (z + h^{(2)})/\epsilon^{(2)}$. At the outer edge of the bottom Stokes boundary layer, the mean Eulerian drift velocity is

$$\overline{u}_2^{(2)} = \tfrac{3}{4}\alpha^2 c_1^2 A^2(T). \quad (4.20)$$

This result is used as a boundary condition for the solution in the core region of the bottom layer at $z = -h^{(2)}$, correct to $O(\alpha^2)$.

4.2. *The core regions*

We now write the velocity $\boldsymbol{q}$ as

$$\boldsymbol{q}^{(r)} = \overline{\boldsymbol{q}}^{(r)} + \tilde{\boldsymbol{q}}^{(r)}, \quad (4.21)$$

where $\overline{\boldsymbol{q}}^{(r)}$ and $\tilde{\boldsymbol{q}}^{(r)}$ are the mean and oscillatory velocity components, respectively. The oscillatory component $\tilde{\boldsymbol{q}}^{(r)}$ can be expanded, in terms of the wave slope α, as

$$\tilde{\boldsymbol{q}}^{(r)} = \tilde{\boldsymbol{q}}_1^{(r)} + \tilde{\boldsymbol{q}}_2^{(r)} + O(\alpha^3), \quad (4.22)$$

where $\tilde{\boldsymbol{q}}_j^{(r)} = O(\alpha^j)$, $j = 1, 2, \ldots$, and the tilde denotes the oscillatory component. In the core regions outside the Stokes boundary layers, the leading-order motions are irrotational wave motions, i.e. $\boldsymbol{q}_1^{(r)} = \tilde{\boldsymbol{q}}_1^{(r)}$. The equations for the oscillatory velocity of $O(\alpha^2)$ are

$$\frac{\partial \tilde{\boldsymbol{q}}_2^{(r)}}{\partial t} + \left(\frac{\beta}{\alpha}\right) \frac{\partial \tilde{\boldsymbol{q}}_1^{(r)}}{\partial T} + \left(\widetilde{\boldsymbol{q}_1^{(r)} \cdot \nabla \boldsymbol{q}_1^{(r)}}\right) = -\nabla \tilde{p}_2^{(r)}, \tag{4.23}$$

$$\nabla \cdot \tilde{\boldsymbol{q}}_2^{(r)} = 0, \tag{4.24}$$

where $O(\alpha) = O(\beta) = O(\epsilon^{(r)})$ has been assumed. Because the first-order solutions are in the form of

$$\tilde{\boldsymbol{q}}_1^{(r)} = A(T) \tilde{\boldsymbol{q}}_1^{(r)} \exp[\mathrm{i}(x - t)], \tag{4.25}$$

$\tilde{\boldsymbol{q}}_2^{(r)}$ can be expressed as, by (4.23),

$$\tilde{\boldsymbol{q}}_2^{(r)} = A(T) \widetilde{\boldsymbol{Q}}_{2,1}^{(r)} \exp[\mathrm{i}(x - t)] + A^2(T) \widetilde{\boldsymbol{Q}}_{2,2}^{(r)} \exp[2\mathrm{i}(x - t)], \tag{4.26}$$

where $\widetilde{\boldsymbol{Q}}_1^{(r)}, \widetilde{\boldsymbol{Q}}_{2,1}^{(r)}$ and $\widetilde{\boldsymbol{Q}}_{2,2}^{(r)}$ are functions of z only. Substituting (4.26) into (4.23) and taking the curl of the resulting equation, we find that $\widetilde{\boldsymbol{Q}}_{2,1}^{(r)}$ and $\widetilde{\boldsymbol{Q}}_{2,2}^{(r)}$ are also irrotational, i.e.

$$\nabla \times \widetilde{\boldsymbol{Q}}_{2,1}^{(r)} = 0, \quad \nabla \times \widetilde{\boldsymbol{Q}}_{2,2}^{(r)} = 0. \tag{4.27}$$

The time-averaged (over a wave period) equation of motion can be obtained, by employing (4.25), (4.26) and (4.27), as

$$\beta \frac{\partial \overline{\boldsymbol{q}}^{(r)}}{\partial T} + \overline{\boldsymbol{Q}}^{(r)} \cdot \nabla \overline{\boldsymbol{Q}}^{(r)} + \nabla \overline{p}^{(r)} - \tfrac{1}{2} \epsilon^{(r)2} \nabla^2 \overline{\boldsymbol{q}}^{(r)} = -\tfrac{1}{2} A^2(T) \nabla \left(\boldsymbol{Q}_1^{(r)*} \cdot \boldsymbol{Q}_{2,1}^{(r)}\right) + O(\alpha^4), \tag{4.28}$$

$$\nabla \cdot \overline{\boldsymbol{q}}^{(r)} = 0, \tag{4.29}$$

where * denotes the complex conjugate. Note that the viscous diffusion in (4.28) must be balanced by the leading order of the inertial force, the first term on the left-hand-side, so that the solutions of the mean motions can be matched with the boundary conditions on the mean levels of the free surface and interface, and at the outer edge of the bottom Stokes boundary layer. Equation (4.28) suggests that just outside the Stokes boundary layers there are thin boundary layers of $O(\epsilon^{(r)} \beta^{-1/2}) = O(\epsilon^{(r)1/2})$. In these second boundary layers viscous diffusion is still important for the mean motions. Away from these second boundary layers in the core regions, the viscous diffusion term in (4.28) can be neglected.

From (4.25) and (4.26), it is clear that the nonlinear forcing term in (4.28) is independent of x. The mean velocity and/or vorticity at the mean levels of the free surface and interface, and at the outer edges of the bottom Stokes boundary layer, (4.6), (4.13), (4.14) and (4.20), are also independent of x. Therefore, we can expect that the mean Eulerian drift velocity in the entire system be uniform in the x-direction. Furthermore the normal component of the mean velocity is zero up to $O(\alpha^2)$ at the mean levels of the free surface and interface, and at the outer edge of the bottom boundary layers, because the free surface, the interface and the bottom boundary are all material surfaces. The vertical component of the mean velocity, $\overline{v}^{(r)}$, must vanish in all the core regions by the continuity equation (4.29) and thus the convection term on the left-hand side of (4.28) disappears. By differentiating the vertical component of (4.28) with respect to x, we obtain

$$\frac{\partial^2 \overline{p}^{(r)}}{\partial z \partial x} = 0. \tag{4.30}$$

Hence, the mean horizontal pressure gradient is constant throughout each of the core regions. Finally the governing equations for the second-order mean Eulerian drift velocity outside the Stokes boundary layers can be simplified as

$$\frac{\partial \overline{u}^{(r)}}{\partial T} - \frac{1}{2}\left(\frac{\epsilon^{(r)2}}{\beta}\right)\frac{\partial^2 \overline{u}^{(r)}}{\partial z^2} = \overline{f}^{(r)}(T), \tag{4.31}$$

$$\overline{v}^{(r)} = 0, \tag{4.32}$$

where $\overline{f}^{(r)}(T)$ denotes the mean horizontal pressure gradient, which is a function of slowly varying time T and depends on the system conditions.

When the system is open and infinitely long in the direction of wave propagation, the mean pressures gradient can be assumed to vanish in the entire system, so that

$$\frac{\partial \overline{p}^{(1)}}{\partial x} = \frac{\partial \overline{p}^{(2)}}{\partial x} = 0. \tag{4.33}$$

When the system is very long but with a closed end, mean pressure gradients are established and so that the net mass flux across the cross-section normal to the direction of wave propagation is zero for each layer of fluid (Longuet-Higgins 1953), i.e.

$$\int_0^{h^{(1)}} u_m^{(1)} \mathrm{d}z = \int_{-h^{(2)}}^{0} u_m^{(2)} \mathrm{d}z = 0, \tag{4.34}$$

where u_m is the mass transport velocity (or Lagrangian velocity) in the core regions, and can be expressed as the total of the mean Eulerian velocity and the Stokes drift, up to $O(\alpha^2)$ (Longuet-Higgins 1953),

$$u_m^{(r)} = \overline{u}^{(r)} + u_s^{(r)} = \overline{u}^{(r)} + \overline{\frac{\partial u_1^{(r)}}{\partial x}\int^t u_1^{(r)} \mathrm{d}t + \frac{\partial u_1^{(r)}}{\partial z}\int^t w_1^{(r)} \mathrm{d}t}. \tag{4.35}$$

The Stokes drift velocities in the core regions are readily obtained by the first-order potential solutions, (3.3) and (3.4),

$$u_s^{(1)} = \tfrac{1}{2}\alpha^2 A^2(T)\left\{\left[1 + \left(\hat{g}\hat{k}/\hat{\sigma}^2\right)^2\right]\cosh 2(z - h^{(1)}) + 2\left(\hat{g}\hat{k}/\hat{\sigma}^2\right)\sinh 2(z - h^{(1)})\right\}, \quad 0 \leqslant z \leqslant h^{(1)}, \tag{4.36}$$

$$u_s^{(2)} = \tfrac{1}{2}\alpha^2 A^2(T) c_1^2 \cosh 2(z + h^{(2)}), \quad 0 \geqslant z \geqslant -h^{(2)}. \tag{4.37}$$

Hence the mean pressure gradients in each layer can be found from the zero-net-flux conditions, (4.34), after the mean Eulerian velocity has been solved.

To find the distribution of the mean flow in the core regions of both the fluid layers, it is necessary to solve (4.31) and (4.32) as an initial-boundary-value problem. The boundary conditions on the mean levels of the free surface and interface, and at the outer edge of the bottom boundary layer are defined by (4.6), (4.15), (4.16) and (4.20), respectively, for $T \geqslant 0$. Since the mean motions in the core regions develop just after the establishment of the Stokes boundary layers, the initial conditions can be defined as

$$\overline{u}^{(1)} = 0 \quad \text{for } T \leqslant 0,\ h^{(1)} > z > 0, \tag{4.38}$$

$$\overline{u}^{(2)} = 0 \quad \text{for } T \leqslant 0,\ 0 > z > -h^{(2)}. \tag{4.39}$$

By (4.16) the mean velocity gradient, $\partial\overline{u}^{(r)}/\partial z$, is of $O(\alpha^2/\epsilon^{(2)})$ at the outer edges of the interfacial Stokes boundary layers. Equation (4.31) shows that adjacent to the interfacial Stokes boundary layers $\partial/\partial z = O(\epsilon^{(r)-1/2})$ for the mean velocity. Therefore, the leading-order mean velocity must be of $O(\alpha^2\epsilon^{(2)-1/2})$. We now express the mean velocity in the following form:

$$\overline{u}^{(r)} = \alpha^2\epsilon^{(2)-1/2}\overline{u}^{(r)}_{3/2} + \alpha^2\overline{u}^{(r)}_2 + o(\alpha^2). \tag{4.40}$$

Hence, the initial-boundary-value problems for the mean motions in the core regions can be summarized as follows:

$O(\alpha^2\epsilon^{(2)-1/2})$

$$\frac{\partial\overline{u}^{(r)}_{3/2}}{\partial T} - \tfrac{1}{2}\left(\frac{\epsilon^{(r)2}}{\beta}\right)\frac{\partial^2\overline{u}^{(r)}_{3/2}}{\partial z^2} = \overline{f}^{(r)}_{3/2}(T), \tag{4.41}$$

with

$$\frac{\partial\overline{u}^{(1)}_{3/2}}{\partial z} = 0 \quad \text{on } z = h^{(1)}, \tag{4.42a}$$

$$\overline{u}^{(1)}_{3/2} = \overline{u}^{(2)}_{3/2} \quad \text{on } z = 0, \tag{4.42b}$$

$$\frac{\partial\overline{u}^{(2)}_{3/2}}{\partial z} - \left(\frac{\hat{\rho}^{(1)}\hat{\nu}^{(1)}}{\hat{\rho}^{(2)}\hat{\nu}^{(2)}}\right)\frac{\partial\overline{u}^{(1)}_{3/2}}{\partial z} = -\tfrac{1}{2}A^2(T)c^{(2)}_2\epsilon^{(2)-1/2}\left(u^{(2)}_0 - u^{(1)}_0\right) \quad \text{on } z = 0, \tag{4.42c}$$

$$\overline{u}^{(2)}_{3/2} = 0 \quad \text{on } z = -h^{(2)}, \tag{4.42d}$$

$$\overline{u}^{(1)}_{3/2} = 0 \quad \text{for } T \leqslant 0,\ h^{(1)} > z > 0, \tag{4.42e}$$

$$\overline{u}^{(2)}_{3/2} = 0 \quad \text{for } T \leqslant 0,\ 0 > z > -h^{(2)}; \tag{4.42f}$$

$O(\alpha^2)$

$$\frac{\partial\overline{u}^{(r)}_2}{\partial T} - \tfrac{1}{2}\left(\frac{\epsilon^{(r)2}}{\beta}\right)\frac{\partial^2\overline{u}^{(r)}_2}{\partial z^2} = \overline{f}^{(r)}_2(T), \tag{4.43}$$

with

$$\frac{\partial\overline{u}^{(1)}_2}{\partial z} = 2A^2(T)\left(\frac{\hat{g}\hat{k}}{\hat{\sigma}^2}\right) \quad \text{on } z = h^{(1)}, \tag{4.44a}$$

$$\overline{u}^{(2)}_2 - \overline{u}^{(1)}_2 = -A^2(T)\left[c^{(2)}_2\left(u^{(2)}_0 + \tfrac{1}{4}c^{(2)}_2\right) + c^{(1)}_2\left(u^{(1)}_0 - \tfrac{1}{4}c^{(1)}_2\right)\right] \text{ on } z = 0, \tag{4.44b}$$

$$\frac{\partial\overline{u}^{(2)}_2}{\partial z} = \left(\frac{\hat{\rho}^{(1)}\hat{\nu}^{(1)}}{\hat{\rho}^{(2)}\hat{\nu}^{(2)}}\right)\frac{\partial\overline{u}^{(1)}_2}{\partial z} \quad \text{on } z = 0, \tag{4.44c}$$

$$\overline{u}^{(2)}_2 = \tfrac{3}{4}c^2_1A^2(T) \quad \text{on } z = -h^{(2)}, \tag{4.44d}$$

$$\overline{u}^{(1)}_2 = 0 \quad \text{for } T \leqslant 0,\ h^{(1)} > z > 0, \tag{4.44e}$$

$$\overline{u}^{(2)}_2 = 0 \quad \text{for } T \leqslant 0,\ 0 > z > -h^{(2)}. \tag{4.44f}$$

4.3. *Solutions*

4.3.1. *Transformed solutions*

The time derivatives in (4.41) and (4.43) can be removed by means of Laplace transform. The transformed variables are

$$\overline{U}^{(r)}_{3/2} = \int_0^\infty \overline{u}^{(r)}_{3/2} e^{-sT} dT, \qquad \overline{F}^{(r)}_{3/2} = \int_0^\infty \overline{f}^{(r)}_{3/2}(T) e^{-sT} dT, \tag{4.45a}$$

$$\overline{U}^{(r)}_{2} = \int_0^\infty \overline{u}^{(r)}_{2} e^{-sT} dT, \qquad \overline{F}^{(r)}_{2} = \int_0^\infty \overline{f}^{(r)}_{2}(T) e^{-sT} dT. \tag{4.45b}$$

It is understood that s is a transform variable hereafter, not the curvilinear coordinate as before. The transformed equations of (4.41) and (4.43) are ordinary differential equations, and can be readily solved analytically with transformed boundary conditions from (4.42*a*)–(4.42*d*) and (4.44*a*)–(4.44*d*). The transformed solutions of the mean velocities are

$$\overline{U}^{(r)}_{3/2} = \frac{1}{s} F^{(r)}_{3/2} + D^{(r)}_3 \exp\left(-\gamma^{(r)} s^{1/2} z\right) + D^{(r)}_4 \exp\left(\gamma^{(r)} s^{1/2} z\right), \tag{4.46}$$

$$\overline{U}^{(r)}_{2} = \frac{1}{s} F^{(r)}_{2} + D^{(r)}_5 \exp\left(-\gamma^{(r)} s^{1/2} z\right) + D^{(r)}_6 \exp\left(\gamma^{(r)} s^{1/2} z\right), \tag{4.47}$$

where $\gamma^{(r)} = (2\beta)^{1/2}/\epsilon^{(r)}$, and $D^{(r)}_3$, $D^{(r)}_4$, $D^{(r)}_5$ and $D^{(r)}_6$ are constants determined by the boundary conditions and the system conditions (see Appendix B).

The transformed forms of the mean Eulerian velocity at the outer edges of the interfacial Stokes boundary layers, (4.13) and (4.14), can be written as

$$\overline{U}^{(r)} = D^{(r)}_1 z - D^{(r)}_2, \tag{4.48}$$

where

$$D^{(r)}_1 = \int_0^\infty d^{(r)}_1 e^{-sT} dT, \qquad D^{(r)}_2 = \int_0^\infty \left(d^{(r)}_2 + \overline{\kappa_1 \varphi^{(r)}_1}\Big|_{z=0} \right) e^{-sT} dT.$$

The transformed solutions, (4.46) and (4.47), in the core regions should match with (4.48) at the outer edges of the interfacial Stokes boundary layers, $z = 0$. By employing the method of matched asymptotic expansions, we obtain

$$D^{(r)}_1 = \gamma^{(r)} s^{1/2} \left[\alpha^2 \epsilon^{(2)-1/2} \left(D^{(r)}_4 - D^{(r)}_3 \right) + \alpha^2 \left(D^{(r)}_6 - D^{(r)}_5 \right) \right], \tag{4.49}$$

$$D^{(r)}_2 = -\alpha^2 \epsilon^{(2)-1/2} \left(\frac{1}{s} F^{(r)}_{3/2} + D^{(r)}_3 + D^{(r)}_4 \right) - \alpha^2 \left(\frac{1}{s} F^{(r)}_{3/2} + D^{(r)}_5 + D^{(r)}_6 \right). \tag{4.50}$$

By (4.49), (4.50), (4.9) and (4.10) , we can conclude that inside the interfacial Stokes boundary layers, where n is of $O(\epsilon^{(r)})$, the leading order of the mean Eulerian velocity is $O(\alpha^2 \epsilon^{(2)-1/2})$ and it is constant across the interfacial boundary layers. It is clear from (4.42*c*) that this leading-order mean velocity is induced by the mean Reynolds stress in the interfacial boundary layers.

4.3.2. *Laplace transform inversion*

To find the physical mean velocity, the inverse transform of solutions (4.46) and (4.47) must be performed. Because an analytical transform inversion of such complicated solutions is impossible, the 'collocation' inversion method of Schapery (1962) is used for the numerical transform inversion. Following Schapery (1962), Rizzo

& Shippy (1970) and Young & Liggett (1977), we approximate the physical mean velocity at a fixed spatial point by a finite Dirichlet series

$$\overline{u}^{(r)}(T) = \sum_{j=1}^{J} a_j^{(r)} e^{-b_j^{(r)}T}, \tag{4.51}$$

where $a_j^{(r)}$ and $b_j^{(r)}$ are constants to be determined. A necessary and sufficient condition for a function to be expandable in a convergent Dirichlet series is that the integral of the square of the function over all positive T exists. In the present analysis, the evolution of the mean velocity in the core regions is assumed to satisfy this condition.

Equation (4.51) is transformed and multiplied by s to yield

$$s\overline{U}^{(r)}(s) = \sum_{j=1}^{J} \frac{a_j^{(r)}}{1+b_j^{(r)}/s}. \tag{4.52}$$

If the $b_j^{(r)}$ are chosen by some means, then J values of s can be selected so that (4.52) represents J equations in the undetermined coefficients, $a_j^{(r)}$. The error can be nearly minimized by choosing $b_j^{(r)}$ to be equal to s_j (Schapery 1962), i.e.

$$b_j^{(r)} = s_j, \quad j = 1, 2, ...J. \tag{4.53}$$

If $s\overline{U}^{(r)}(s)$ versus $\log s$ is plotted, the significant range of s needed in the inversion scheme is where $s\overline{U}^{(r)}(s)$ shows a definite variation. Numerical experience (Schapery 1962; Rizzo & Shippy 1970; Young & Liggett 1977) has shown that optimal results are achieved by selecting the s_j in a geometric sequence, i.e. $s_{j+1}/s_j = \lambda$, where λ is a fixed ratio. The upper and lower bounds of s_j are selected from the plot and the ratio, λ, is fixed by choosing J, or *vice versa.* The accuracy is generally increased by choosing a large J. Thus J simultaneous linear equations in a_j are formed:

$$\sum_{j=1}^{J} \frac{a_j^{(r)}}{1+s_j/s_m} = s_m\overline{U}^{(r)}(s_m), \quad m = 1, 2, ...J. \tag{4.54}$$

4.3.3. *Numerical results and discussion*

In order to check our analysis and numerical method, we first simulate a single-layer problem by letting $\hat{\rho}^{(2)}/\hat{\rho}^{(1)} = 10^{12}, \epsilon^{(1)} = \epsilon^{(2)} = 0.001, h^{(1)} = h^{(2)} = 1, \overline{f}^{(1)}(T) = \overline{f}^{(2)}(T) = 0$ in the present solutions. In this system the mean Reynolds stress vanishes at the outer edges of the interfacial Stokes boundary layers. Therefore the mean velocity of $O(\alpha^2\epsilon^{(2)-1/2})$ does not exist and the leading-order mean velocity is of $O(\alpha^2)$. Figure 1(*a*) shows the typical significant range for s by plotting $s\overline{U}_2^{(1)}$ versus s at $z = 0.02$, which is located in the interfacial second boundary layer. The significant range for s is clearly around $O(1)$. The parameters for the numerical Laplace transform inversion are $J = 20, \lambda = 2, s_J = 256$. A standard Gauss–Jordan elimination procedure is used to solve the resulting linear equations in $a_j^{(1)}$, (4.54). The result yields a continuous solution in time T for the mean Eulerian velocity $\overline{u}_2^{(1)}$ at each spatial point. The numerical inversion procedure is repeated for each vertical position in the core region of the top fluid layer. The resulting velocities in the bottom layer are very small and are neglected. The mean Eulerian velocity $\overline{u}_2^{(1)}$ at the outer edge of the interfacial Stokes boundary layer, $z = 0$, in the top layer, is shown in figure 1(b). It is decaying with time by a factor $\exp(-2T)$. The mean Eulerian velocity profiles in the core region of the top layer, outside the Stokes boundary layers, are shown in figure 1(c) for $T = 0.5, 1, 2$, where $\bar{u}_2^{(1)}$ has been scaled by the initial mean velocity at $z = 0$. The mean velocity or vorticity at the outer edges of the Stokes boundary layers, induced by the first-order wave motions and viscosity, begins to diffuse into the core region at the on-set of the mean motions. The second boundary layers are formed near the free surface and interface, adjacent to the Stokes boundary layers. Because of viscous attenuation of the mean velocity or vorticity source at the outer edges of the Stokes boundary layers and viscous diffusion of the mean velocity, the second boundary layers diminish as time T increases. As it diffuses into the centre of the core region, the mean velocity becomes significantly small. The analytical solutions of $\bar{u}$ obtained by Craik (1982) for a single-layer fluid system with zero mean pressure gradient are also shown in figure 1(b) and 1(c) for comparison. The present numerical solutions show excellent agreement with the analytical solutions.

To examine the transient problem of the mean motions in the core regions of a two-layer fluid system, we set $\hat{\rho}^{(2)}/\hat{\rho}^{(1)} = 2, \hat{\nu}^{(2)}/\hat{\nu}^{(1)} = 2, \epsilon^{(1)} = 0.001, h^{(2)} = h^{(1)} = 1$. In the present analysis, a small-amplitude interfacial wave, which corresponds the minus sign in the dispersion relation (A 6), is assumed to be propagating in the fluids. The following values are chosen for the parameters of the numerical Laplace transform inversion, $J = 20, \lambda = 2, s_J = 128$. The same numerical inversion procedure as that for the single-layer system is used to calculate the mean Eulerian velocities in both the layers. Figure 2 shows the results for a open system with zero mean pressure gradients. Figure 2(a, b) presents the evolution of the mean Eulerian velocity at the outer edges of the interfacial Stokes boundary layers. The mean velocity of $O(\alpha^2)$, $\bar{u}_2$, vanishes when $T > 2$. But the mean velocity of $O(\alpha^2\epsilon^{(2)-1/2})$, $\bar{u}_{3/2}$, persists much longer and disappears after $T > 40$. The resulting mean Eulerian velocity profiles in the core regions are shown in figure 2(c, d), at different time T. Strong second boundary layers are established adjacent to the interfacial and bottom Stokes boundary layers. The mean motions are mainly restricted to these second boundary layers. The mean velocity and vorticity diffuse into the core regions as time increases owing to viscosity. But because the mean velocity and vorticity sources at the outer edges of the Stokes boundary layers decay due to viscous attenuation, the mean velocities in the centres of the core regions are significantly smaller. The mean velocity of $O(\alpha^2\epsilon^{(2)-1/2})$ is induced by the interfacial mean stresses. Therefore, second boundary layers for $\bar{u}_{3/2}$ exists only adjacent to the interfacial Stokes boundary layers. The leading-order mean motions exists long ($T > 20$) after the wave motions vanish ($T > 2$).

The numerical results for mass transport velocity in a closed system are shown in figure 3. The same values as those in an open system are used for the numerical inversion parameters. The evolution of the mean velocity at the outer edges of interfacial Stokes boundary layers is similar to that in an open system. Second boundary layers for the mean motions are established adjacent to the Stokes boundary layers. The mean velocities in the centres of core regions are enhanced by viscous diffusion and the mean pressure gradients. The leading-order mean velocity persists longer and dies out eventually.

The dominant mean Eulerian velocities in a two-layer system, $O(\alpha^2\epsilon^{-1/2})$, are much larger than that in a single-layer system, $O(\alpha^2)$, and persist much longer because of the gradual diffusion of the interfacial velocity and vorticity of $O(\alpha^2\epsilon^{-1/2})$ from the interface into the core regions. The second boundary layer near the free surface is significantly weaker in a two-layer system.

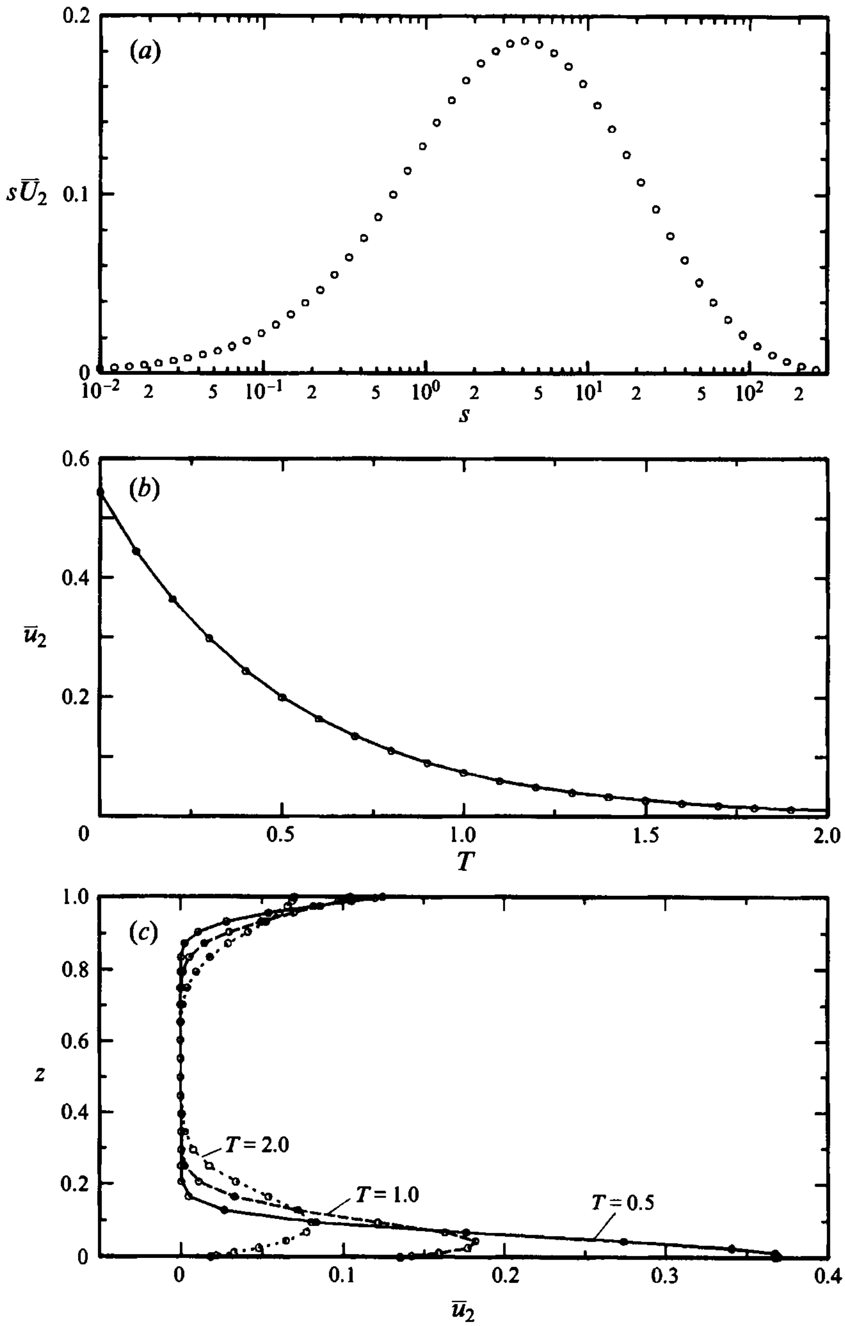

FIGURE 1. An open system with $\hat{\rho}^{(2)}/\hat{\rho}^{(1)} = 10^{12}$, $\epsilon^{(2)} = \epsilon^{(1)} = 0.001$, $h^{(1)} = h^{(2)} = 1$: (*a*) Transformed mean velocity on $z = 0.02$; (*b*) mean velocity $\bar{u}_2$ at the outer edge of the interfacial Stokes boundary layer, $z = 0$, in the top layer; (*c*) mean velocity $\bar{u}_2$ in the core region of the top layer. ○ ○ ○ ○ ○, The numerical solutions, and lines are for the analytical solutions (Craik 1982).

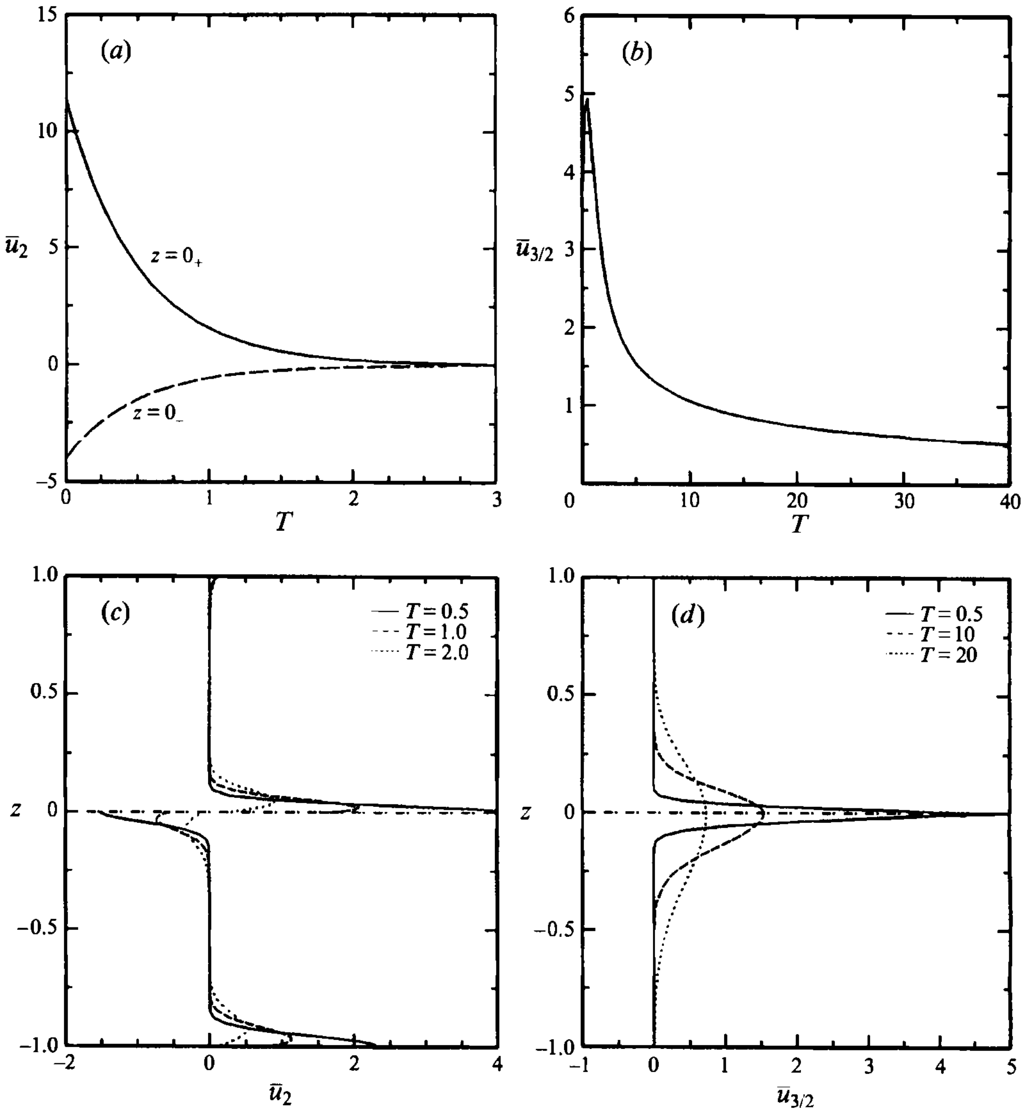

FIGURE 2. Mean Eulerian velocity of an interfacial wave in an open system with $\hat{\rho}^{(2)}/\hat{\rho}^{(1)} = 2, \hat{\nu}^{(2)}/\hat{\nu}^{(1)} = 2, \epsilon^{(1)} = 0.001, h^{(1)} = h^{(2)} = 1$. (a) $\bar{u}_2$ on $z = 0$; (b) $\bar{u}_{3/2}$ on $z = 0$; (c) $\bar{u}_2$ in the core regions; (d) $\bar{u}_{3/2}$ in the core regions.

4.4. *Mean motions of surface waves*

There exist two possible modes for wave motions in a two-layer fluid system, i.e. the surface wave mode and the internal wave mode, corresponding to the $\pm$ signs used in the dispersion relation (A 6). With the surface wave mode, the interfacial wave amplitude is smaller than that of the free surface, while the interfacial wave amplitude is larger in the internal wave mode motions. Up to now the analysis has been focused on the mean motions associated with an internal wave mode. It is also of interest to examine the mean velocities induced by wave motions of a surface wave mode in a two-layer fluid system with viscous attenuation.

When a surface wave is propagating in a two-layer fluid system, the interfacial displacement is smaller than that on the free surface, and the difference of potential velocity across the interface becomes smaller. Therefore the rotational velocities in the interfacial Stokes boundary layers, which exist to ensure continuity of the tangential velocity and stress across the interface, are smaller. The interfacial Stokes boundary layers are much weaker than that of an interfacial wave. The magnitudes of the potential velocity difference across the interface and of the rotational velocity in the lower interfacial Stokes boundary layer on the interface $n = 0$ can be expressed as, by (3.3), (3.4), and (3.15),

$$\left|\frac{\partial \varphi_1^{(2)}}{\partial z} - \frac{\partial \varphi_1^{(1)}}{\partial z}\right|_{z=0} = \alpha A(T)\left|u_0^{(2)} - u_0^{(1)}\right|,$$

$$\left|\frac{\partial \chi_1^{(2)}}{\partial n}\right|_{n=0} = \alpha A(T)\left|c_2^{(2)}\right|.$$

With the following typical values of wave parameters: $\hat{\rho}^{(2)}/\hat{\rho}^{(1)} = 2, \hat{v}^{(2)}/\hat{v}^{(1)} = 2, \epsilon^{(1)} = 0.001, h^{(2)} = 0.5$, figure 4($a$) shows $\left(u_0^{(2)} - u_0^{(1)}\right)$ and $c_2^{(2)}$ versus $h^{(1)}$. The leading-order rotational velocities in the interfacial Stokes boundary layers are no longer of $O(\alpha)$ like that of an interfacial wave. Therefore the mean Reynolds stress, induced by the surface wave motions inside the interfacial Stokes boundary layer, is much smaller. The jumps of the mean velocity and of the mean velocity gradient across the interfacial Stokes boundary layers are of $O(\alpha^2)$ and $O(\alpha^2\epsilon^{(2)-1/2})$ respectively, as shown in figure 4(b), i.e.

$$\mathrm{D}\bar{u} = \left(\bar{u}^{(2)} - \bar{u}^{(1)}\right)\big|_{z=0} = O(\alpha^2), \tag{4.55}$$

$$\mathrm{D}\left(\frac{\partial \bar{u}}{\partial z}\right) = \left[\frac{\partial \bar{u}^{(2)}}{\partial z} - \left(\frac{\hat{\rho}^{(1)}\hat{v}^{(1)}}{\hat{\rho}^{(2)}\hat{v}^{(2)}}\right)\frac{\partial \bar{u}^{(1)}}{\partial z}\right]\bigg|_{z=0} = O(\alpha^2\epsilon^{(2)-1/2}). \tag{4.56}$$

Inside the second boundary layers adjacent to the Stokes boundary layers, $\partial/\partial z = O(\epsilon^{(r)-1/2})$. So there is no driving force for the mean motions of $O(\alpha^2\epsilon^{(2)-1/2})$ for a surface wave and the mean velocity of $O(\alpha^2\epsilon^{(2)-1/2})$ vanishes in the whole system as can be concluded from (4.41) and (4.42). Because all the forcing terms for the mean motions in the core regions are from the boundary conditions at the outer edges of the Stokes boundary layers, (4.6), (4.20), (4.55) and (4.56), and correspond to the mean velocity of $O(\alpha^2)$, the leading-order mean motions of a surface progressive wave must be of $O(\alpha^2)$. The mean Eulerian velocity of a surface progressive wave can be readily obtained by solving the problem consisting of (4.43) and (4.44) with (4.44c) being replaced by

$$\frac{\partial \bar{u}_2^{(2)}}{\partial z} - \left(\frac{\hat{\rho}^{(1)}\hat{v}^{(1)}}{\hat{\rho}^{(2)}\hat{v}^{(2)}}\right)\frac{\partial \bar{u}_2^{(1)}}{\partial z} = -\tfrac{1}{2}A^2(T)\frac{c_2^{(2)}}{\epsilon^{(2)}}\left(u_0^{(2)} - u_0^{(1)}\right) \quad \text{on } z = 0. \tag{4.57}$$

The solution procedures are similar to those for the mean motions of an interfacial wave as described in § 4.3. With details of the solution procedures being neglected, we present some numerical results in figure 4(c–f). The mean velocities at the outer edges of the interfacial Stokes boundary layers become significantly small when $T > 2$. The mean velocity profiles are similar to those of the mean velocities of $O(\alpha^2)$ for an interfacial wave. Second boundary layers exists near the free surface, interface and bottom boundary, adjacent to the Stokes boundary layers. The mean motions in the system decay as T increases.

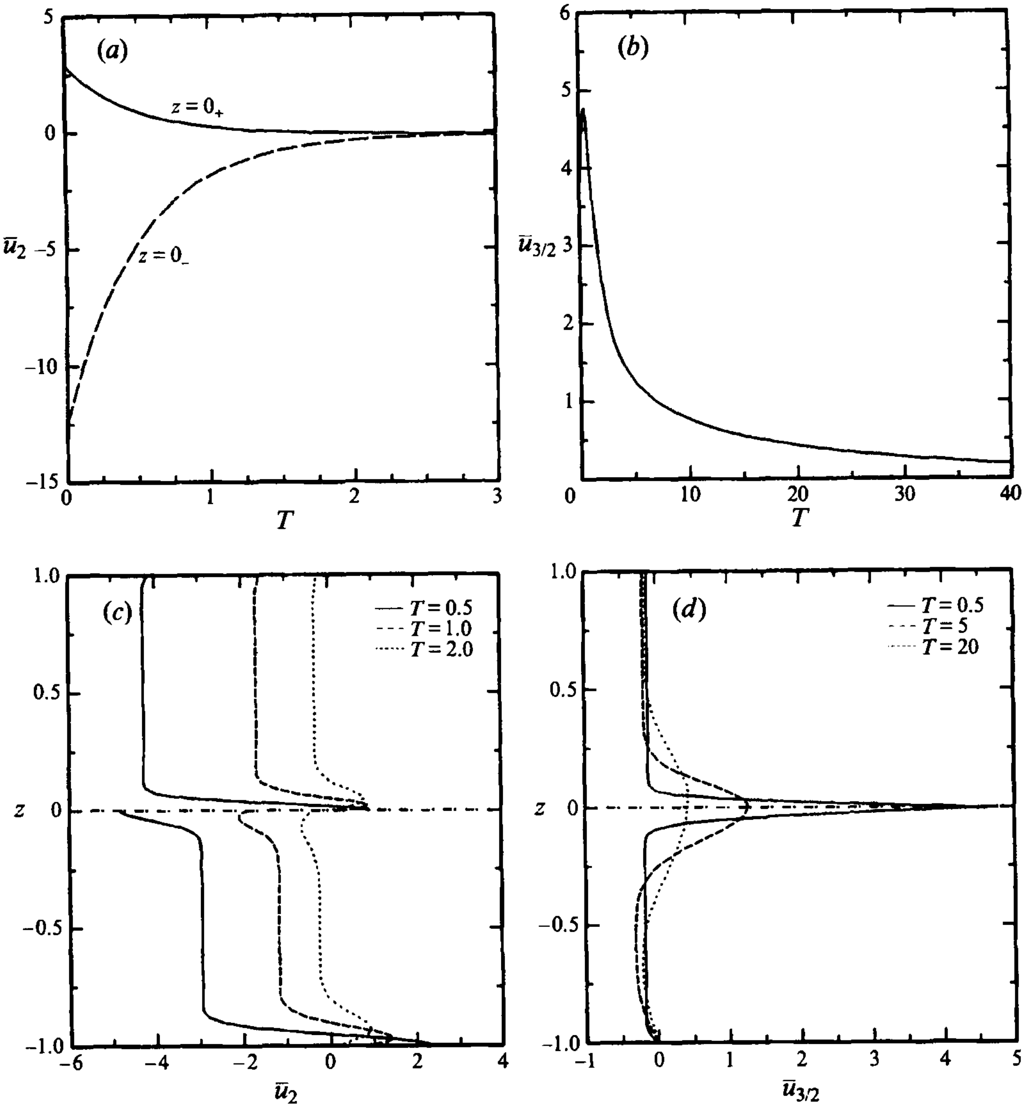

FIGURE 3. As figure 2 but for a closed system.

5. Concluding remarks

The effects of viscous attenuation on the mass transport velocity (or the mean Eulerian drift velocity) associated with an interfacial progressive wave, which is assumed to decay temporally due to viscosity, have been studied in a two-layer fluid system. The interface is considered to be immiscible and discontinuous in fluid properties. It is found that the strong interfacial mean Reynolds stresses are induced by the first-order wave motions. As mean interfacial velocity and vorticity diffuse into the core regions, outside the Stokes boundary layers, after the on-set of the wave motions, second boundary layers are formed near the interface, adjacent to the interfacial Stokes boundary layers. A strong steady streaming of $O(\alpha^2\epsilon^{-1/2})$ exists in the second boundary layers. Because of viscous attenuation and vertical diffusion,

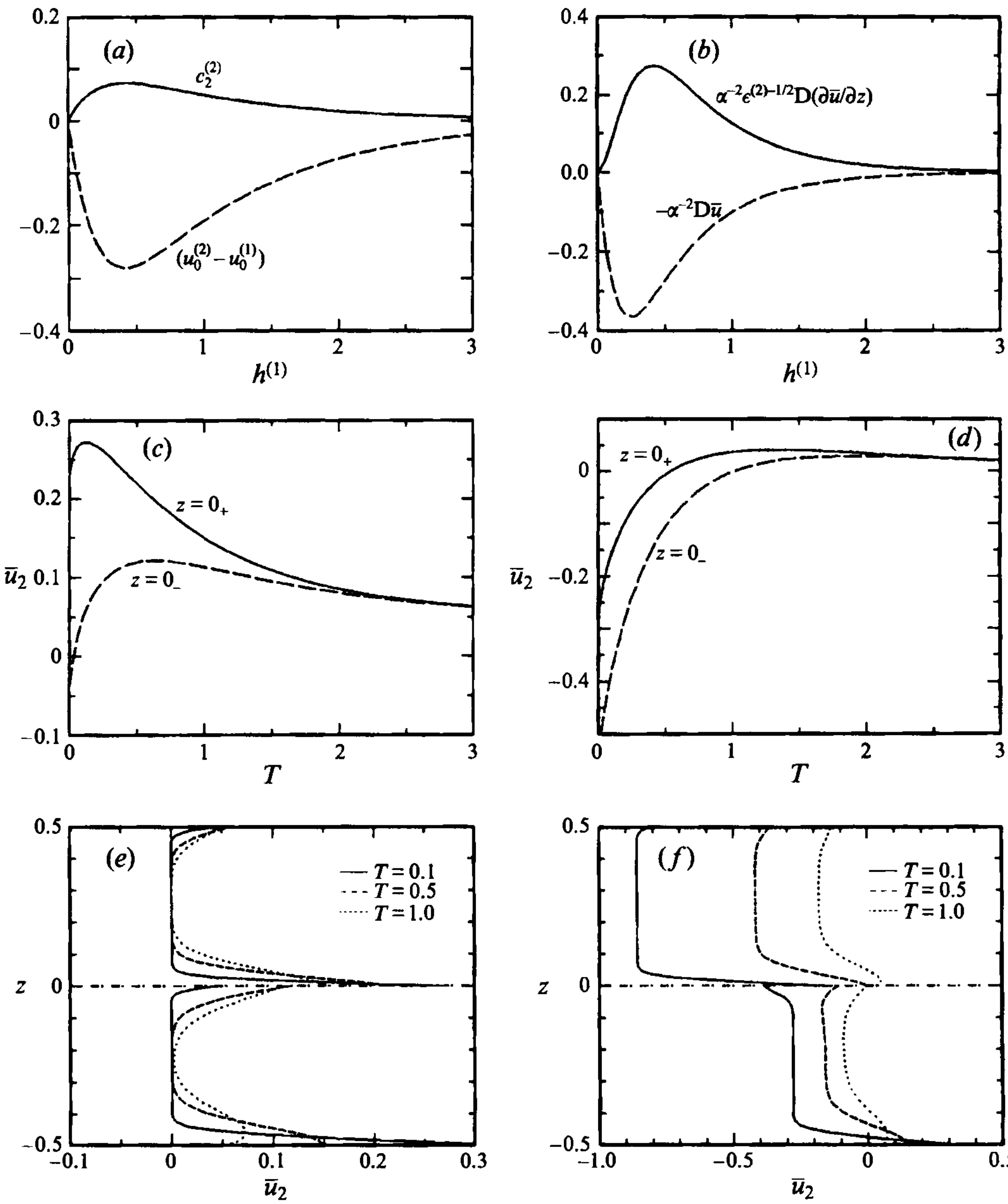

FIGURE 4. A surface wave with $\hat{\rho}^{(2)}/\hat{\rho}^{(1)} = 2, \hat{\nu}^{(2)}/\hat{\nu}^{(1)} = 2, \epsilon^{(1)} = 0.001$: (*a*) $c_2^{(2)}$ and $(u_0^{(2)} - u_0^{(1)})$ with $h^{(2)} = 0.5$; (*b*) $\mathrm{D}\left(\partial\bar{u}/\partial z\right)$ and $\mathrm{D}\bar{u}$ at $T = 0$ with $h^{(2)} = 0.5$; (*c*) $\bar{u}_2$ on $z = 0$ in an open system with $h^{(1)} = h^{(2)} = 0.5$; (*d*) $\bar{u}_2$ on $z = 0$ in a closed system with $h^{(1)} = h^{(2)} = 0.5$; (*e*) $\bar{u}_2$ in the core regions of an open system with $h^{(1)} = h^{(2)} = 0.5$; (*f*) $\bar{u}_2$ in the core regions of a closed system with $h^{(1)} = h^{(2)} = 0.5$.

the second boundary layers vanish as time increases. The mass transport velocities in the centres of the core regions are enhanced by the strong mean interfacial vorticity. The mean motions in the whole system are stronger and persist longer than that in a single-layer fluid system. The results are different from the solutions obtained by Dore (1970, 1973) who neglected the effects of viscous damping. Dore found that the mass transport velocity is of $O(\alpha^2\epsilon^{-1})$ in the entire two-layer system and its vertical profiles are parabolic. Because $O(\epsilon) \ll 1$, the mass transport velocity without viscous attenuation is much larger than our result, which is $O(\alpha^2\epsilon^{-1/2})$. However, Dore's analysis is restricted to the condition $O(\alpha) \ll O(\epsilon)$. When the wave amplitude is of the same order of magnitude as the Stokes boundary layer thickness, $O(\alpha) \approx O(\epsilon)$, Dore's analysis breaks down because his mass transport velocity has the same order magnitude as the leading-order wave motions.

When a surface progressive wave is propagating in a two-layer fluid system, the leading-order mean velocity is of $O(\alpha^2)$, much smaller than that of an interfacial wave, $O(\alpha^2\epsilon^{-1/2})$. The mean velocity profiles are similar to those of mean motions of $O(\alpha^2)$ of an interfacial progressive wave. Transient second boundary layers develop adjacent to the Stokes boundary layers.

Viscous attenuation of the mass transport velocity associated with temporally decaying waves in a single-layer fluid system was studied as a quasi-steady problem by Liu & Davis (1977), whose solution exhibits some anomalous singularities. Craik (1982) re-examined the problem and treated it as an initial-boundary-value problem. The solution obtained by Liu & Davis (1977) was then proved to be a particular solution to the problem. The singularities were resolved. In the present study, the mass transport associated with a temporally decaying interfacial progressive wave in a two-fluid system has been studied as an initial-boundary-value problem in a similar way to Craik for a single-layer fluid system. Unlike a single-layer system, however, it is impossible to obtain an analytical solution of the mass transport velocity in a two-layer system because the mean motions in the two layers are coupled through the interfacial boundary conditions. Therefore a Laplace transform with a numerical inversion has been used to solve the problem. This approach is proved to be effective.

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Appendix A. The leading-order solutions

$$c_1 = \frac{\cosh h^{(1)}}{\sinh h^{(2)}} 1 - (\hat{g}\hat{k}/\hat{\sigma}^2)\tanh h^{(1)}, \tag{A 1}$$

$$c_2^{(1)} = -\frac{(\hat{\rho}^{(2)}/\hat{\rho}^{(1)})(\hat{\nu}^{(2)}/\hat{\nu}^{(1)})^{1/2}\cosh h^{(1)}}{1+(\hat{\rho}^{(2)}/\hat{\rho}^{(1)})(\hat{\nu}^{(2)}/\hat{\nu}^{(1)})^{1/2}} \times \left\{ \left[1-\left(\hat{g}\hat{k}/\hat{\sigma}^2\right)\tanh h^{(1)}\right]\coth h^{(2)} + \tanh h^{(1)} - \left(\hat{g}\hat{k}/\hat{\sigma}^2\right) \right\}, \tag{A 2}$$

$$c_2^{(2)} = c_2^{(1)}/[(\hat{\rho}^{(2)}/\hat{\rho}^{(1)})(\hat{\nu}^{(2)}/\hat{\nu}^{(1)})^{1/2}], \tag{A 3}$$

$$u_0^{(1)} = -\sinh h^{(1)} + (\hat{g}\hat{k}/\hat{\sigma}^2)\cosh h^{(1)}, \qquad u_0^{(2)} = c_1\cosh h^{(2)}. \tag{A 4}$$

The interfacial displacement is

$$\xi_1 = A(T)\cosh h^{(1)}[1-(\hat{g}\hat{k}/\hat{\sigma}^2)\tanh h^{(1)}]\exp[\mathrm{i}(x-t)], \tag{A 5}$$

and the dispersion relation is

$$\frac{\hat{\sigma}^2}{\hat{g}\hat{k}} = \frac{1}{2M}\left[-L \pm (L^2-4MN)^{1/2}\right], \tag{A 6}$$

where

$$L = -\left(\hat{\rho}^{(2)}/\hat{\rho}^{(1)}\right)\tanh(h^{(1)}+h^{(2)})\left(1+\tanh h^{(1)}\tanh h^{(2)}\right),$$

$$M = \left(\hat{\rho}^{(2)}/\hat{\rho}^{(1)}\right) + \tanh h^{(1)}\tanh h^{(2)}, \quad N = \left[\left(\hat{\rho}^{(2)}/\hat{\rho}^{(1)}\right)-1\right]\tanh h^{(1)}\tanh h^{(2)}.$$

Appendix B. The transformed mean velocity

B.1. *Coefficients used in an open system*

$$D_3^{(1)} = \frac{E_4}{E_5}\mathrm{e}^{G^{(1)}}\sinh G^{(2)}, \quad D_4^{(2)} = \frac{E_4}{E_5}\mathrm{e}^{G^{(2)}}\cosh G^{(1)}, \tag{B 1}$$

$$D_4^{(1)} = D_3^{(1)}\mathrm{e}^{-2G^{(1)}}, \qquad D_3^{(2)} = -D_4^{(2)}\mathrm{e}^{-2G^{(2)}}, \tag{B 2}$$

$$D_5^{(1)} = \frac{1}{E_5}\left[E_1 R\sinh G^{(2)} - E_1\cosh G^{(2)} + E_2\mathrm{e}^{G^{(1)}} - E_3\mathrm{e}^{G^{(1)}}\cosh G^{(2)}\right], \tag{B 3}$$

$$D_6^{(2)} = \frac{1}{E_5}\left[RE_1\mathrm{e}^{G^{(2)}} + E_2\cosh G^{(1)} - E_2R\sinh G^{(1)} + RE_3\mathrm{e}^{G^{(2)}}\sinh G^{(1)}\right], \tag{B 4}$$

$$D_6^{(1)} = E_1\mathrm{e}^{-G^{(1)}} + D_5^{(1)}\mathrm{e}^{-2G^{(1)}}, \qquad D_5^{(2)} = E_2\mathrm{e}^{-G^{(2)}} - D_6^{(2)}\mathrm{e}^{-2G^{(2)}}, \tag{B 5}$$

where

$$G^{(1)} = \gamma^{(1)}h^{(1)}s^{1/2}, \qquad G^{(2)} = \gamma^{(2)}h^{(2)}s^{1/2}, \tag{B 6}$$

$$R = \frac{\hat{\rho}^{(1)}\epsilon^{(1)}}{\hat{\rho}^{(2)}\epsilon^{(2)}}, \qquad E_1 = \frac{2}{\gamma^{(1)}}\left(\frac{\hat{g}\hat{k}}{\hat{\sigma}^2}\right)\frac{1}{s^{1/2}(s+2)}, \qquad E_2 = \frac{3}{4}\frac{c_1^2}{s+2}, \tag{B 7}$$

$$E_3 = -\frac{1}{s+2}\left[c_2^{(2)}\left(u_0^{(2)} + \tfrac{1}{4}c_2^{(2)}\right) + c_2^{(1)}\left(u_0^{(1)} - \tfrac{1}{4}c_2^{(1)}\right)\right], \tag{B 8}$$

$$E_4 = -\frac{1}{2}\frac{c_2^{(2)}\epsilon^{(2)-1/2}}{s^{1/2}\gamma^{(2)}(s+2)}\left(u_0^{(2)} - u_0^{(1)}\right). \tag{B 9}$$

$$E_5 = \left[(1+R)\cosh\left(G^{(1)}+G^{(2)}\right) + (1-R)\cosh\left(G^{(1)}-G^{(2)}\right)\right], \tag{B 10}$$

B.2. *Coefficients used in a closed system*

$$D_3^{(1)} = \frac{E_4}{F_9}(F_4F_1 - F_3F_2)\mathrm{e}^{G^{(1)}}, \quad D_4^{(1)} = D_3^{(1)}\mathrm{e}^{-2G^{(1)}}, \tag{B 11}$$

$$\frac{1}{s}\overline{F}_{3/2}^{(1)} = \frac{1}{G^{(1)}}\left[D_4^{(1)}\left(1-\mathrm{e}^{G^{(1)}}\right) - D_3^{(1)}\left(1-\mathrm{e}^{-G^{(1)}}\right)\right], \tag{B 12}$$

$$D_4^{(2)} = -\frac{F_1}{F_9}E_4(F_6\mathrm{e}^{-G^{(1)}} + F_5\mathrm{e}^{G^{(1)}}), \tag{B 13}$$

$$D_3^{(2)} = -\frac{F_2}{F_1}D_4^{(2)}, \quad \frac{1}{s}\overline{F}_{3/2}^{(2)} = -D_3^{(2)}\mathrm{e}^{G^{(2)}} - D_4^{(2)}\mathrm{e}^{-G^{(2)}}, \tag{B 14}$$

$$D_5^{(1)} = \frac{1}{F_9}\Big\{(F_4F_1 - F_3F_2)RE_1 + (F_7+E_2)(F_3+F_4)\mathrm{e}^{G^{(1)}} + (F_1+F_2)\left[F_6E_1 - (F_8+E_3)\mathrm{e}^{G^{(1)}}\right]\Big\} \tag{B 15}$$

$$D_6^{(1)} = E_1\mathrm{e}^{-G^{(1)}} + D_5^{(1)}\mathrm{e}^{-2G^{(1)}}, \tag{B 16}$$

$$\frac{1}{s}\overline{F}_2^{(1)} = (F_8 - F_7) + \frac{1}{G^{(1)}}\left[D_6^{(1)}\left(1 - e^{G^{(1)}}\right) - D_5^{(1)}(1 - e^{-G^{(1)}})\right], \tag{B 17}$$

$$D_6^{(2)} = -\frac{1}{F_9}\left\{2R\sinh G^{(1)}\left[F_3(F_7 + E_2) - F_1(F_8 + E_3)\right] + RE_1F_1(F_5 + F_6) + (F_7 + E_2)\left(F_6e^{-G^{(1)}} + F_5e^{G^{(1)}}\right)\right\}, \tag{B 18}$$

$$D_5^{(2)} = \frac{1}{F_1}\left(F_7 + E_2 - F_2D_4^{(2)}\right), \quad \frac{1}{s}\overline{F}_2^{(2)} = E_2 - D_5^{(2)}e^{G^{(2)}} - D_6^{(2)}e^{-G^{(2)}}, \tag{B 19}$$

where

$$F_1 = e^{G^{(2)}} + \frac{1}{G^{(2)}}\left(1 - e^{G^{(2)}}\right), \quad F_2 = e^{-G^{(2)}} - \frac{1}{G^{(2)}}\left(1 - e^{-G^{(2)}}\right), \tag{B 20}$$

$$F_3 = 1 + \frac{1}{G^{(2)}}\left(1 - e^{G^{(2)}}\right), \quad F_4 = 1 - \frac{1}{G^{(2)}}\left(1 - e^{-G^{(2)}}\right), \tag{B 21}$$

$$F_5 = -1 + \frac{1}{G^{(1)}}\left(1 - e^{-G^{(1)}}\right), \quad F_6 = -1 - \frac{1}{G^{(1)}}\left(1 - e^{G^{(1)}}\right), \quad F_7 = \frac{c_1^2\sinh 2h^{(2)}}{4h^{(2)}(s+2)}, \tag{B 22}$$

$$F_8 = \frac{c_1^2\sinh 2h^{(2)}}{4h^{(2)}(s+2)} - \frac{1}{4h^{(1)}(s+2)} \times\left\{\left[1 + \left(\hat{g}\hat{k}/\hat{\sigma}^2\right)^2\right]\sinh 2h^{(1)} + 2\left(\hat{g}\hat{k}/\hat{\sigma}^2\right)\left(1 - \cosh 2h^{(1)}\right)\right\} \tag{B 23}$$

$$F_9 = -(F_1 + F_2)\left(F_6e^{-G^{(1)}} + F_5e^{G^{(1)}}\right) + 2R\sinh G^{(1)}(F_4F_1 - F_3F_2). \tag{B 24}$$

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